Generalized Noiseless Quantum Codes utilizing Quantum Enveloping Algebras

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Abstract

A generalization of the results of Rasetti and Zanardi concerning avoiding errors in quantum computers by using states preserved by evolution is presented. The concept of dynamical symmetry is generalized from the level of classical Lie algebras and groups to the level of a dynamical symmetry based on quantum Lie algebras and quantum groups (in the sense of Woronowicz). A natural connection is proved between states preserved by representations of a quantum group and states preserved by evolution with dynamical symmetry of the appropriate universal enveloping algebra. Illustrative examples are discussed.

The dynamical symmetry of a system is a valuable property. One can apply to systems which possess a dynamical symmetry powerful methods based on the theory of Lie algebras and their representations, like the method of coherent states [1]. Dynamical symmetry has also proved to be important in searching for physical systems with very specific quantum states: states which can not be corrupted by their interactions with the environment [2]. These states can be used to provide noiseless quantum codes that have great potential utility for constructing quantum computers. Noiseless quantum codes can be an alternative or a supplement to error correcting codes, which are elaborate methods for coding information, recognizing errors and correcting them [3, 4, 5].

This paper introduces the notion of a dynamical symmetry associated with quantum groups. We apply this concept of dynamical symmetry to the study of systems which demonstrate the usefulness of noiseless quantum codes.

We first outline the basic mathematical concepts and tools that will be used in the paper. Quantum groups will be understood as C^*-Hopf algebras following [6, 7]. However, in our further considerations only the *-Hopf algebra structure of quantum groups is used; the C^*-algebra structure is not important in our applications.

A *-Hopf algebra is a complex unital algebra $\mathcal{A}$ (with the unit $1_{\mathcal{A}} \in \mathcal{A}$) equipped with linear maps of the coproduct $\Phi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, counit $e : \mathcal{A} \to \mathbb{C}$ and the antipode $\kappa : \mathcal{A} \to \mathcal{A}$, for which the following identities hold:

$$
\begin{aligned}
(\mathrm{id}_{\mathcal{A}} \otimes \Phi)\Phi &= (\Phi \otimes \mathrm{id}_{\mathcal{A}})\Phi \\
(e \otimes \mathrm{id}_{\mathcal{A}})\Phi(a) &= (\mathrm{id}_{\mathcal{A}} \otimes e)\Phi(a) = a \\
m(\kappa \otimes \mathrm{id}_{\mathcal{A}})\Phi(a) &= m(\mathrm{id}_{\mathcal{A}} \otimes \kappa)\Phi(a) = e(a)1_{\mathcal{A}}
\end{aligned}
\qquad (1)
$$

where $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the product in A, explicitly given by $m(a \otimes b) = a \cdot b$. We also assume that there is an antilinear involutive map $^*: \mathcal{A} \to \mathcal{A}$ which is antimultiplicative in the sense that $(ab)^* = b^* a^*$, and which is compatible with Φ in the sense that $\Phi^* = (^* \otimes ^*)\Phi$.

The whole classical theory of Lie groups and Lie algebras can be viewed as a special case of the theory of quantum groups. For classical groups, the algebra $\mathcal{A}$ will be commutative and consist of complex-valued polynomial functions over the group. Furthermore, in the classical case * is the standard complex conjugation of functions, and the maps Φ, e and κ represent the product in the group, the neutral element, and the operation of taking inverses of the elements of the group, respectively. The algebraic identities (1) correspond in the classical case to the associativity of the group product (coassociativity of Φ), multiplication of the element with its inverse giving the neutral element (1), and multiplication of an element with the unit element giving original element [7]. General noncommutative *-Hopf algebras $\mathcal{A}$ have no such interpretation, but are considered informally as algebras of 'functions' on more general objects called *quantum groups* G.

A generalization of the concept of dynamical symmetry can be defined only when there are well-established notions of a Lie algebra and a corresponding universal enveloping algebra associated with each quantum group G, including the corresponding representation theory. In quantum theory, all these notions depend essentially on an appropriately chosen *differential calculus* over G.

First-order differential calculi are defined as certain *modules* Γ over $\mathcal{A}$, equipped with a *differential* $d: \mathcal{A} \to \Gamma$. The module Γ is a noncommutative counterpart of the usual module of 1-forms over a classical group, and d generalizes the standard differential of functions.

In quantum group theory, a special role is played by so-called *left-covariant* and *bicovariant* differential calculi. In these cases, we can introduce the analogs of left and left/right actions of the group G on the module [8]. If the module Γ is left-covariant, then we can define a subspace Γ_{inv} of Γ, which consists of left-invariant '1-forms'. A quantum Lie algebra can then be defined as the corresponding dual space, $L = \Gamma^*_{\mathrm{inv}}$.

If the calculus is bicovariant, then we can introduce a natural *braid operator* $\sigma : L \otimes L \to L \otimes L$, playing the role of the classical transposition. Furthermore, in analogy with classical theory, we can define the Lie bracket in the space L [8]. The Lie bracket is defined by an operator $C : L \otimes L \to L$, as $[\cdot,\cdot] : L \times L \to L$, $[x, y] = C(x \otimes y)$, and it satisfies an appropriately generalized Jacobi identity.

Following classical theory, the quantum universal enveloping algebra for $(L, [,])$ is defined as a unital associative algebra $U(L)$ generated by the relations $xy - \sum_i y_i x_i = [x, y]$, where $x, y \in L$ and $\sum_i y_i \otimes x_i = \sigma(x \otimes y)$.

Having this bracket and using the above equation one can define representations of quantum Lie algebras and of the corresponding quantum universal enveloping algebras. It can be shown that every representation v of G in a finite-dimensional vector space V naturally gives rise to a representation $S : U(L) \to \mathrm{End}(V)$ of the quantum universal enveloping algebra. Namely, let $v : V \to V \otimes \mathcal{A}$ be a (left) representation of the quantum group (*-Hopf algebra) $\mathcal{A}$ in a finite dimensional complex vector space V (i.e. v is linear and satisfies the conditions $(\mathrm{id}_V \otimes \Phi)v = (v \otimes \mathrm{id}_{\mathcal{A}})v$, and $(\mathrm{id}_V \otimes e)v = \mathrm{id}_V$, corresponding to the usual requirements for representations of groups — the products of group elements are represented by compositions of operators representing these elements, and the neutral element of a group is represented by the identity operator). Every such representation of G in V naturally generates a representation $\delta : U(L) \to \mathrm{End}(V)$ of $U(L)$ in V (if the differential calculus is bicovariant) or only of the Lie algebra L, $\delta : L \to \mathrm{End}(V)$ (if the differential calculus is left-covariant).

Moreover, if the differential calculus is *-covariant, which means that in the module Γ of 1-forms is defined the * -operation $^* : \Gamma \to \Gamma$ induced by * in $\mathcal{A}$, it makes sense to speak about the hermiticity of the representation δ. Namely, the *-operation on Γ naturally induces the *-structure on the quantum Lie algebra L, via the formula $< f^*, \psi >= - < f, \psi^* >$ where $f \in L = \Gamma^*_{\mathrm{inv}}$ and $\psi \in \Gamma^*_{\mathrm{inv}}$. Moreover, if the quantum group is "connected" in the sense that $\ker(d) = \mathbb{C} \cdot 1_{\mathcal{A}}$, then one can prove that these two conditions are equivalent.

Suppose then, that a *-covariant, left-covariant differential calculus is defined on a quantum group G and $L, U(L), V, v, \delta$ are as above. We define an open system with quantum dynamical symmetry as a system whose evolution is defined in the Hilbert space V. The system interacts with its environment described by the Hilbert space, H_B (assumed for simplicity to be finite-dimensional; however, everything could be incorporated into the infinite-dimensional case).

We say that a system has quantum dynamical symmetry described by the quantum group G and its quantum Lie algebra L if the following conditions are satisfied:

i) The evolution of the system is governed by the Hamiltonian

$$h \in \mathrm{End}(V \otimes H_B) \simeq \mathrm{End}(V) \otimes \mathrm{End}(H_B).$$

ii) The Hamiltonian is Hermitian ($h^* = h$) with * defined as the tensor product of natural star operators described above acting in $\mathrm{End}(V)$ and $\mathrm{End}(H_B)$.

iii) The Hamiltonian has the form:

$$h = P_1(l_1, \dots, l_n) \otimes T_1 + \dots + P_N(l_1, \dots, l_n) \otimes T_N \tag{2}$$

where $P_1, \dots, P_N$ are polynomial expressions of infinitesimal generators $l_i = \delta(e_i)$; $\{e_i\}$ is a basis in L; and $T_1, ..., T_N$ are Hermitian operators acting in H_B, $T_\alpha \in H_B$.

Systems with quantum dynamical symmetry can be explored by generalized methods known from the theory of systems possessing a classical dynamical symmetry, e.g. by the method of quantum coherent states [9]. Let us observe that the terms can be reorganized in such a way that the Hamiltonian has the familiar form: $h = h_S + h_B + h_I$, where h_S is the Hamiltonian of such system (sum of terms with $T_i = \mathrm{id}_{H_B}$), h_B is the Hamiltonian of the environment (sum of the terms with constant parts of $\delta(P_i)$ which are id_V) and h_I is the interaction Hamiltonian.

Let $v : V \to V \otimes \mathcal{A}$ be an arbitrary representation of G in the finite-dimensional vector space V, and let $\delta : L \to \mathrm{End}(V)$ be the associated representation of L. To further simplify the considerations, we shall consider the case in which the quantum group is 'connected' in the sense that $ker(d) = \mathbb{C}1_{\mathcal{A}}$.

Then for every vector $u \in V$, $v(u) = u \otimes 1_{\mathcal{A}}$ is equivalent to $\forall x \in L : \delta(x)u = 0$.

Let us now assume that the calculus Γ is also bicovariant. This enables us to introduce the quantum universal enveloping algebra $U(L)$, and to discuss the representations of $U(L)$ associated with the representations of G. Let us introduce the map $\chi : U(L) \to \mathbb{C}$, with the properties $\chi(L) = 0$, $\chi(1) = 1$, and we extend it to $U(L)$ by multiplicativity. The representation δ uniquely (as in the standard theory) extends from L to $U(L)$. The above two conditions are equivalent to $\delta(q)u = \chi(q)u, \quad \forall q \in U(L)$

Vectors satisfying any of the above conditions are called *v-invariant*. Having such v-invariant vectors and an open system with quantum dynamical symmetry one can prove:

Theorem 1

The unitary evolution described by the Hamiltonian h of the form (2) preserves the v-invariance of the vectors and associated states of the system, even when all other states of the system are corrupted due to decoherence.

Proof: Let us take as an initial vector $u \otimes \zeta \in V \otimes H_B$, where u is v-invariant in the sense defined above. Then the unitary evolution defined by: $U(t) = \exp(-\frac{i}{\hbar}ht)$ gives:

$$\exp(-\frac{i}{\hbar}ht)(u \otimes \zeta) = u \otimes \exp(-\frac{i}{\hbar}h_{\text{eff}}t)\zeta \tag{3}$$

$$\text{where} \quad h_{\text{eff}} = \chi(P_1)T_1 + \dots + \chi(P_N)T_N \qquad \square \tag{4}$$

Now we can easily generalize Theorems 1 and 2 given in [2]. We follow the notation of [2]. $\rho_S \in \mathrm{End}(V)$ and $\rho_B \in \mathrm{End}(H_B)$ are states of the system and the environment (bath), respectively. If the

overall system is initially in the state $\rho(0) = \rho_S \otimes \rho_B$, then $\rho(t) = U(t)\rho(0)U(t)^+$, so that the evolution is unitary. The induced evolution on V is, similarly to that in [2], given by $L_t^{\rho_B} : \rho_S \to tr^B \rho(t)$, where tr^B is the trace over H_B. Then the following theorem can be proved:

Theorem 2

Let $\mathcal{M}_N$ be the manifold of states built over the space of vectors invariant under the representation v, and $\rho_S \in \mathcal{M}_N$. Then for any initial bath state ρ_B the induced evolution on V is trivial, $L_t^{\rho_B}(\rho(t)) = \rho$, $\forall t > 0$.

Theorem 1 allows us to reduce the proof of Theorem 2 to the proof of the first theorem of [2].

The invariant vectors are generalizations of the singlet states in [2] as the states of the quantum register which are not corrupted by the interaction with the environment.

Before we present simple examples illustrating the general theory and explicitly demonstrating "error-protected" states, let us discuss the interesting question of the structure of the Hilbert space of the registers of the quantum computer. We will also discuss the physical implications. The register usually consists of a number of copies of the same quantum system, often having two possible states, e.g. *spin up* and *spin down* (qubit).

Dynamical symmetry acts in the Hilbert space that originates from the Hilbert space for an individual qubit being described as a representation space of our quantum group G: $v_i : V_i \to V_i \otimes \mathcal{A}$, $i = 1, \ldots, n$. The Hilbert space is the tensor product of the representation spaces, $V = V_1 \otimes V_2 \otimes \ldots \otimes V_n$ on which the tensor product of representations v_i, $v = v_1 \times v_2 \times \ldots \times v_n$ acts. Since with each of the representations v_i is associated a representation δ_i of the corresponding quantum universal enveloping algebra, to the representation v corresponds the representation δ of the quantum universal enveloping algebra. One can prove [8] the following relation (for $n = 2$):

$$\delta(x)(\phi_1 \otimes \phi_2) = \sum_\alpha \delta_1(x^\alpha)\phi_1 \otimes \phi_2^\alpha + \phi_1 \otimes \delta_2(x)\phi_2 \tag{5}$$

where $\tau(\phi_2 \otimes x) = \sum_\alpha x^\alpha \otimes \phi_2^\alpha$ and $\tau : V_2 \otimes L \to L \otimes V_2$ is the appropriate *flip-over* operator uniquely defined by the differential calculus.

This formula differs from the corresponding formula for the classical case [10] of addition of angular momenta in quantum mechanics (τ in the classical case is just the standard transposition). Its diagrammatic representation

Fig. 1 (6)

and its generalization to arbitrary n-fold coupling

Fig 2 (7)

show that the qubits in the register are not treated on the same footing. It could be associated to some effects due to, not taken into account [2], linear extension of the register, or to fluctuations

of the fields due to nonideal structure of boundaries of the register and their influence, etc. It is possible then to realize a system with weaker symmetry than the one presented in [2]. It is known that similar deviations from exact dynamical symmetry of Lie groups lead to better mass or energy formulas in nuclear, particle, and molecular physics, e.g. [11], [12]. Therefore, one can look for possible candidates for registers of quantum computers there.

In [2] the physically plausible conjecture was expressed that small deviations from ideal properties of the system should lead to small errors in the error-protected states. Actually, we have shown that there exist systems with special kind of deviation from the assumed symmetry, which nevertheless still have error-protected states.

Let us examine some simple examples that illustrate our general ideas and theorems. The first example of a quantum group presented systematically in the literature was the $S_\mu U(2)$ group [13], where the C^*-algebraic approach to quantum groups was used. In [13] not only the algebraic and functional analytic aspects were treated, but also the geometry of $S_\mu U(2)$, including the left-covariant three-dimensional calculus, and the bicovariant four-dimensional calculus (discussed also in detail in [14]). Generalization of the results concerning this particular quantum group leads to the general theory of compact matrix quantum groups [6, 7], to the definition of quantum spheres [15] and their geometry [16], to deep generalization of the Tannaka-Krein duality [8], and also to the theory of quantum principal bundles together with the corresponding gauge theory on quantum spaces, first formulated in [17] and then developed systematically in [18, 19] (see also [20, 21, 22]). Also in the C^*-algebraic framework, quantum homogeneous bundles were defined and the example of such a bundle with quantum spheres as fibers was given [23]. Simultaneously, a different approach to quantum groups was developed by Soviet [24, 25] and Japanese schools [26], in which quantum groups are treated from the point of view of deformations of the universal enveloping algebras. In the latter approach, the utilization of quantum groups for studying completely solvable systems seems to be the main motivation for developing the theory.

In this paper, we conceptually follow the first of these approaches. We use the quantum group $S_\mu U(2)$ in our examples. First, we recall some basic facts about $S_\mu U(2)$ (the case $\mu \in [-1, 1]$, $\mu = 1$ corresponds to the classical $SU(2)$ group). This quantum group is based on a *-algebra $\mathcal{A}$ generated by elements α, α^*, γ, γ^* satisfying the following relations:

$$\alpha\alpha^* + \mu^2\gamma^*\gamma = 1_{\mathcal{A}},\ \ \alpha^*\alpha + \gamma^*\gamma = 1_{\mathcal{A}},\ \ \gamma^*\gamma = \gamma\gamma^*,\ \ \alpha\gamma = \mu\gamma\alpha,\ \ \alpha\gamma^* = \mu\gamma^*\alpha. \tag{8}$$

The comultiplication, counit, and the antipode are defined on the generators of the algebra by:
i) comultiplication:
$\Phi(\alpha) = \alpha \otimes \alpha - \mu\gamma^* \otimes \gamma, \quad \Phi(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$
with $\Phi(\alpha^*), \Phi(\gamma^*)$ fixed by the property $\Phi^* = (^*\otimes^*)\Phi$.
ii) counit: $e(\alpha) = 1 \quad e(\gamma) = 0$
iii) antipode: $\kappa(\alpha) = \alpha^*, \quad \kappa(\alpha^*) = \alpha, \quad \kappa(\gamma) = -\mu\gamma, \quad \kappa(\gamma^*) = (-1/\mu)\gamma^*$

From the point of view of our examples, the theory of representations of $S_\mu U(2)$ is very interesting. The theory has many similarities to the theory of representations of $SU(2)$. Since the representations (also in the case of a general quantum group G) are linear maps $v : V \to V \otimes \mathcal{A}$, we can introduce a matrix representation in a given basis $\{e_i\}$ in V, by $v(e_i) = \sum_j e_j \otimes v_{ji}$. In the case in which the basis is orthonormal and v is unitary, the matrix of the representation v is also unitary (in the extended sense in which the Hermitian conjugate of a matrix is the transposition of the matrix created by *-conjugation of its elements). As a basic example of such representation, consider the fundamental representation of $S_\mu U(2)$, defined in an orthonormal basis by the matrix:

$$u_{ij} = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix}. \tag{9}$$

The reader can easily check using the defining relations for $S_\mu U(2)$ that the matrix is unitary:$u^*u = uu^* = 1$, where the unit on the right side is actually the tensor product of the unit in the $S_\mu U(2)$ with the unit 2×2 matrix.

In our considerations it is essential that the irreducible unitary representations of $S_\mu U(2)$ are associated with half-integers, like the representations of $SU(2)$. The fundamental representation introduced by (9) corresponds to $j = \frac{1}{2}$. The Clebsch-Gordan decompositions of tensor products of the representations of the $S_\mu U(2)$ into irreducible representations are similar (concerning the multiplicities of the appearence of irreducible components in the products of representations) to the classical case:

$$u^{\otimes n} = \bigoplus_{j \in J} n_j u_j \tag{10}$$

In particular, the decomposition of the second tensor power of the fundamental representation is $u_{1/2}^{\otimes 2} = u_0 \oplus u_1$, where u_0 and u_1 are the 1-dimensional and the 3-dimensional irreducible representations, respectively. One can describe these representations more explicitly after introducing an orthonormal basis in the representation space $V = \mathbb{C}^2$ of $u_{1/2}$, which will be denoted $|+\rangle$, $|-\rangle$ for the purpose of being familiar to physicists. The tensor product $u_{1/2}^{\otimes 2}$ is realized in $V \otimes V \simeq \mathbb{C}^4$, and the orthonormal basis in this space is $|+\rangle \otimes |+\rangle$, $|+\rangle \otimes |-\rangle$, $|-\rangle \otimes |+\rangle$, $|-\rangle \otimes |-\rangle$. It is an easy exercise to show that the invariant subspaces of $u_{1/2}^{\otimes 2}$ are spanned by:

$$\frac{1}{\sqrt{1+\mu^2}}(|+\rangle \otimes |-\rangle - \mu|-\rangle \otimes |+\rangle) \tag{11}$$

$$|+\rangle \otimes |+\rangle, \qquad \frac{\mu}{\sqrt{1+\mu^2}}(|+\rangle \otimes |-\rangle + \frac{1}{\mu}|-\rangle \otimes |+\rangle), \qquad |-\rangle \otimes |-\rangle \tag{12}$$

(11) generalizes the singlet state, and (12) generalizes the triplet state. In analogy to the classical case, the even tensor powers of the fundamental representation decompose into irreducible represen-

tations in such a way that the one-dimensional representation appears a number of times, and the number is identical to the classical case. These singlets are preserved by the dynamics.

Examples:

1) In the first example we treat a system which is as close as possible to the one considered in [2]. Namely, as a model of the environment (bath) we consider a system of harmonic oscillators, described by the Hamiltonian $h_B = \sum_k \omega_k b_k^\dagger b_k$, acting in the Hilbert space H_B, $h_B \in \mathrm{End}(H_B)$. The register consists in this simplest case of two qubits. In contrast to the case considered by Zanardi and Rasetti [2], the system consisting of the register and the bath has the dynamical symmetry not of the $SU(2)$ group but of the $S_\mu U(2)$ quantum group. As previously mentioned, in the quantum group context it is necessary to choose a differential calculus, prior to establishing the notion of the dynamical symmetry associated with a given quantum group. The closest calculus to the classical case seems to be the $3D$ left-covariant calculus [13]. In other words, the quantum Lie algebra L is 3-dimensional. Let us denote by K_i the operators representing the basis vectors l_i, in an arbitrary representation of L (here $i \in \{1,2,3\}$). The following recurrent formulas enable us to compute explicitly the operators K_i, in the arbitrary tensor product of elementary 2-dimensional representations – qubits (where $j \in \{1,2\}$):

$$\begin{aligned} K_3(\psi \otimes |+\rangle) &= \frac{1}{2}\psi \otimes |+\rangle + \frac{1}{\mu^2} K_3(\psi) \otimes |+\rangle && (13) \\ K_3(\psi \otimes |-\rangle) &= \mu^2 K_3(\psi) \otimes |-\rangle - \frac{1}{2}\psi \otimes |-\rangle && (14) \\ K_j(\psi \otimes |+\rangle) &= \frac{1}{2}\psi \otimes |+\rangle + \frac{1}{\mu} K_j(\psi) \otimes |+\rangle && (15) \\ K_j(\psi \otimes |-\rangle) &= \mu K_j(\psi) \otimes |-\rangle - \frac{1}{2}\psi \otimes |-\rangle, && (16) \end{aligned}$$

In this case, the bath-register interaction Hamiltonian which is the quantum group analog of the Hamiltonian used in [2] ($h_I = \sum_k g_k S^+ b_k + f_k S^- b_k^+ + h_k S^z b_k + H.c.$) is: $h_I = K_+ T + K_- T^\dagger + K_3 T'$, where $K_\pm = K_1 \pm i K_2$, and T, T' are operators acting in the bath Hilbert state-space. The operators T and T' are obtained as appropriate linear combinations of the creation and annihilation operators (b_k, b_k^+) describing relevant elementary excitations of the bath. The operators K_j act in the 4-dimensional 2-qubit space. K_+, K_- coresspond to classical S^+, S_-, K_3 corresponds to S^z. In other words, the Hamiltionial is formally the same form as in [2]. However the 'spin' operators are different as explained above. It is obvious that the singlet state of the register is error-protected in the sense discussed above.

2) In the second example the only change from example 1) is the register consists of any even number of qubits, instead of just two. The spin operators K_j refer to the total register system, and are calculated by applying the above rules inductively.

It is important to mention that the number of singlet states is the same as in the classical $SU(2)$ case. This is a consequence of the similarity between the representation theories for quantum and classical $SU(2)$ groups. The dimension of the singlet state space depends on the number of qubits in the way described in [2]. All of these states are clearly protected from corruption due to decoherence.

Coupling of the qubits to the same environment gives more error-protected states than coupling to independent environments [28]. We made this assumption in our examples, as did the authors of [2]. Nevertheless, our methods are general enough to deal with the cases of coupling to independent environments as well when the system has the dynamical symmetry of the type introduced in this paper. After completion of the paper the authors found the paper [27] which extends [2], still in the context of the classical group dynamical symmetry.

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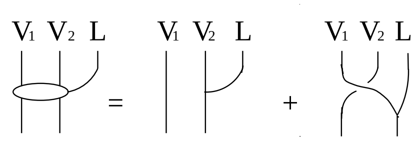
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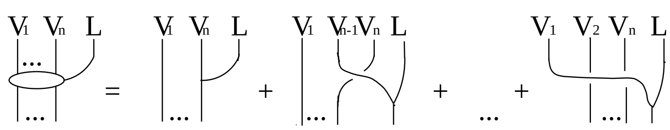

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